\(\in\mu\)-logics as a theory of propositions

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Abstract

An explicit treatment of the inherent ideas or senses of formal expressions beyond the classical truth values in the semantics of a logic comes with many advantages, for example in the field of a formal reconstruction of natural language semantics. In this paper we study the explicit logical interpretation of formal expressions of logics as propositions. In the general understanding propositions refer to the semantical entities which represent the sense (in the understanding of Frege) or state of affairs denoted by a formal expression. We present the class of \(\in\mu\)-logics (read Epsilon-mu-logics) as first defined by Bab in [1] as a special class of propositional logics\(^1\) with means of expression for self-reference, classical connectives, quantification over propositions, and the ability to integrate modalities coming from arbitrary modal logics over possible world semantics. We study the different understandings of the notion of propositions in the literature and argue that an explicit treatment of propositions in logics like \(\in\mu\)-logics naturally is not possible without making further restricting assumptions to the range of propositions like for example the assumption of a referencability of propositions by formal expressions. We show how \(\in\mu\)-logics can be newly interpreted as a theory of propositions by providing the capability for the meta-level reasoning about the models of underlying arbitrary object-level logics and an explicit interpretation of formulas as propositions.

1 Introduction

The term proposition is used in three different understandings in the fields of linguistics, philosophy, logic, and mathematics: in the first understanding propositions refer

\(^1\)Our notion of propositional logics differs from the traditional understanding of such logics. In order to distinguish between these understandings we refer to the traditional understanding by the notion of “traditional propositional logic” and to the understanding of the present work by propositional logics in italic letters.
to syntactical entities like for example formulas of a given logic. In the second understanding (which is more common in modern logic) propositions refer to the inherent \( \text{sense} \)\(^2\) of a given entity which can be either true or false. In the third understanding propositions refer to for example a stated theorem which is to be proven. In this work we use the term of propositions solely in the second understanding. However, the given definition of propositions as senses of certain entities is not sufficient to cover the entire conception of propositions. Many researches have taken place in the fields of propositions since Aristoteles which have resulted in different conceptions and definitions of the notion. Here it becomes obvious that the term of proposition cannot be defined adequately without considering the term of \( \text{judgments} \) at the same time. Questions in the literature on propositions are for example if propositions are syntactical or semantical entities (a discussion on the problems of an ambiguous use of propositions as syntactical and semantical entities was given by Carnap in [9]) or if there is a clear distinction between propositions and judgments at all. We will discuss certain understandings of the term of propositions and give a definition of the understanding of the term in the sense of this work in Section 2. As we will see a proposition in the modern understanding of the notion (as the inherent sense of an entity) is usually defined as being independent from syntactical concepts. We will argue that a formal treatment of propositions in logics naturally cannot be accomplished without further assumptions like for example a referencability of propositions by formulas.

Thus in the field of logic a proposition can be interpreted as the inherent sense of a formula. Normally the inherent senses of formal expressions are not explicitly available in logics. Consider for example the formula

\[ p \land q \]

of traditional propositional logic (where \( p \) and \( q \) are variables and \( \land \) is the connective of conjunction). Here the assertion which shall be represented by the variables \( p \) and \( q \) is fully unconsidered in the syntax and semantics of the logic. From the viewpoint of propositions in the understanding of this paper it can be said that in classical logics the \( \text{senses} \) of formulas (that means the propositions denoted by these formulas) are limited to propositions representing the truth values of being \textit{true} and \textit{false}. For many applications of logic – especially in the field of the reconstruction of natural language semantics – we regard this limitation as a major disadvantage. This is due to the fact that in natural languages one often has different sentences which refer to the same sense or sentences of the same truth value which are completely independent to each other from the viewpoint of their inherent senses. Consider for example the following three formulas of first-order-predicate logic\(^3\) over the standard signature of natural numbers:

\[ \varphi = \text{Prime}(x) \]
\[ \psi = \text{successor}(x) = y \]
\[ \chi = \text{add}(x, \text{one}) = y \]

\(^2\)In this paper we use the term of the \textit{sense} of an entity in the understanding of Frege as its inherent idea (see [15]).

\(^3\)See for example [12] for an introduction into first-order predicate logic.
Consider for example a logical structure of natural numbers in which $Prime$ is being interpreted as the predicate of being a prime number, $successor$ as the operation assigning to each natural number its direct successor, $add$ as the summation of two given natural numbers, and $one$ as the natural number 1. Let further be given an assignment which assigns to $x$ the natural number of 7 and to $y$ the natural number of 8. In this setting the three formulas above do have the same extension under the given logical structure in the semantics of first-order-predicate logic, namely that of being true. The inherent idea (that means the sense) of the formulas, however, is only implicitly present in the semantics of the logic, but not explicitly given as an accessible object. This is due to the fact that the semantics of classical logics are extensional and not intensional. From an intensional point of view it is more natural to interpret the formulas $\psi$ and $\chi$ as intensionally equal as they refer to the same sense by denoting the proposition of adding 1 to the value assigned to $x$. Contrary to that $\phi$ denotes the proposition that the value assigned to $x$ is only divisible by 1 and itself.

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The explicit treatment of propositions in the semantics of a logic comes with a number of advantages and applications which form the goals of this paper:

1. An explicit interpretation of formal expressions as propositions allows for the comparison and distinction of formal expressions on a sense level which is far superior to the semantical classification of expressions solely as the truth values true and false.

2. A concept for the extension of existing logics by an explicit propositional layer allows for defining complex and expressive logics for the meta-level reasoning about underlying given object-levels.

3. Such a meta-level logic is much more capable to cover the semantics of natural language than it is possible in classical logics without an explicit propositional layer. Here the capability of representing natural language semantics increases proportional with the expressiveness of the logic.

4. A class of expressive meta-level propositional logics can be interpreted as a theory of propositions by the entirety of logics and models.

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4. A class of expressive meta-level propositional logics can be interpreted as a theory of propositions by the entirety of logics and models.

In this paper we present the class of $\mathcal{E}_{\mu}$-logics which was first defined by Bab in [1] and show how this class of logics can be newly interpreted as a theory of propositions. $\mathcal{E}_{\mu}$-logics allow for the extension of arbitrary underlying object-level logics by

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4From a certain point of view one could argue that the truth values of being true and false can be interpreted as propositions themselves and that the only propositions denoted by the formulas of classical logics are true and false. That argumentation, however, would be contrary to the view of Frege who made a clear distinction between the sense of a sentence and its truth value. In this paper we conform to the view of Frege. However, in the semantics of $\mathcal{E}_{\mu}$-logics as to be presented in the following sections we consider models in which the formulas of the logics can only be interpreted as the two propositions for true and false. This idea does not contradict to the view of Frege, as there still is a clear distinction between the propositions true and false and the truth values true and false in the semantics of $\mathcal{E}_{\mu}$-logics.

5It should be mentioned here that the given interpretation of the formulas as propositions is only meant as one example showing the advantages of treating the inherent sense of formal expressions explicitly. However, the propositions denoted by formal expressions always depend on a certain point of view under which the expressions are considered. This implies that the term of proposition is depending on some kind of subject, a question which we will pick up further in Section 2.

6See [2] for a short introduction into this work in English language.
the explicit interpretation of the formulas of these object-level logics as propositions. $\varepsilon_\mu$-logics offer means of expression for self-reference, classical connectives, quantification over propositions, the comparison and distinction of propositions denoted by formulas, and the ability to integrate modalities coming from arbitrary modal logics over possible world semantics. As a consequence $\varepsilon_\mu$-logics can be used for reproducing certain natural language semantics (especially in the context of self-referential sentences) in the formal language of a logic and for propositionally reasoning about underlying object-level logics on a meta-level.

The presented work on $\varepsilon_\mu$-logics proceeds and extends two previous works on propositional logics by Werner Sträter and Philipp Zeitz. The nowadays so-called classical $\varepsilon_T$-logic by Werner Sträter (see [36]) introduces a theory of truth and propositions and was first defined in the context of reconstructing natural language semantics by means of self-referential structures (see [29, 26] and [3]). The formulas of classical $\varepsilon_T$-logic are built from classical connectives and come along with a concept of quantification over propositions, means of expression for propositional equality and propositional predicates for truth and falsity. Classical $\varepsilon_T$-logic is intensional in the way that formulas are not only interpreted as true or false, but explicitly interpreted as propositions in the models of classical $\varepsilon_T$-logic. Sträter showed that classical $\varepsilon_T$-logic is free from antinomies despite its total truth-predicates and its ability to model self-referential sentences and impredicative quantification.

Zeitz generalized the concepts of classical $\varepsilon_T$-logic to allow for the extension of arbitrary underlying logics by the concepts of truth, reference, and quantification over propositions. Zeitz’ parameterized $\varepsilon_T$-logic (see [39]) allows for the treatment of formulas of the underlying logics as being propositional constants on the $\varepsilon_T$-logic level. To cover the extension of arbitrary logics in parameterized $\varepsilon_T$-logic Zeitz studied different forms of abstractions of logics and introduced the concept of logics in abstract form, in which the semantics of a logic is given by a system of sets of formulas, which (from some point of view) can be interpreted as the theories of the logic. As Sträter before, Zeitz showed the existence of special intensional models of parameterized $\varepsilon_T$-logic and showed that parameterized $\varepsilon_T$-logic is free from antinomies.

The concept of $\varepsilon_\mu$-logics by Bab (see [1]) as to be presented and re-interpreted in this paper generalizes and gives a new interpretation to the concepts of classical and parameterized $\varepsilon_T$-logic. $\varepsilon_\mu$-logics form a class of propositional logics which extend arbitrary logics in abstract form. In $\varepsilon_\mu$-logics the underlying logic is seen as the object-level logic. The $\varepsilon_\mu$-logic which extends the object-level logic can then be used for the meta-level reasoning about the propositions denoted by the formulas of the underlying logic. Like classical and parameterized $\varepsilon_T$-logic before, any $\varepsilon_\mu$-logic contains means of expression for reference and even self-reference, quantification over propositions, and classical connectives. Furthermore, $\varepsilon_\mu$-logics allow for the integration of modalities coming from arbitrary modal logics over Kripke semantics. Like the $\varepsilon_T$-logics of Sträter and Zeitz before $\varepsilon_\mu$-logics have been proven to be free from antinomies by showing the existence of certain extensional and intensional models. Furthermore Bab showed that the class of $\varepsilon_\mu$-logics fully encompasses classical and parameterized $\varepsilon_T$-logics. In [4] Bab and Wieczorek widely extended the theory of models for classical $\varepsilon_T$-logic by showing the existence of specific intensional models generated from arbitrary equivalence relations on the formulas of classical $\varepsilon_T$-logic which are consistent.
with the truth functionalities of the connectives of $\in_T$-logic. The results of this paper appear to be expandable to $\in_\mu$-logics, too, enabling $\in_\mu$-logics to offer a much wider theory of intensional models.

Further works in the field of $\in_T$-logics are for example the work of Batyuk (see [6]) who extended $\in_\mu$-logics by epistemic concepts, enabling the resulting class of logics to state subject-related meta-level assertions. Another logic in the family of $\in_T$-logics was defined by Lewitzka in [25]. His $\in_T$-logic offers a non-Fregean intuitionistic logic with a truth predicate and a falsity predicate as intuitionistic negation. $\in_T$-logic can be seen as an extension of Sträter's and Zeitz work, but without using the concepts of quantification. Furthermore Lewitzka introduces a new connective that expresses references between statements and thus yields a finer characterization of intensional models.

One key aspect of all logics in the family of $\in_T$-logics – apart from Lewitzka's $\in_T$-logic which targets on other questions – is that all the logics offer means of expression for self-referential and impredicative sentences without losing the existence of a total truth predicate. The problem with reference and especially self-reference is a topic of wide interest in logic which occurs whenever a logic contains means of expression for self-reference together with classical negation and the possibility to speak about the truth and falsity of formal expressions. In such a setting antinomies might occur. Nevertheless all mentioned $\in_T$-logics are free from antinomies and have total truth predicates, but nevertheless offer the possibility to implicitly refer to antinomies by non-satisfiable equations. This is due to a special understanding of referential expressions in these kinds of logics.

Self-reference comes in two different ways in $\in_T$-logics, on the one hand by the use of the operator of propositional identity $\equiv$ for formulating self-referential expressions. Consider for example a formula $\varphi \equiv \psi$ where $\varphi$ shall be a sub-formula of $\psi$. The formula $\varphi \equiv \psi$ states that $\varphi$ and $\psi$ are interpreted as the same proposition. Due to the fact that $\varphi$ is a sub-formula of $\psi$ the formula $\varphi \equiv \psi$ is a self-referential expression on $\varphi$. On the other hand self-reference is given using the quantification over propositions. As in every model of $\in_T$-logics every formula is explicitly interpreted as a proposition, it is also the case that the formula $\forall x. \varphi$ denotes a certain proposition. In the semantics of $\in_T$-logics however, the universal quantifier $\forall$ ranges over all available propositions in the considered model and thus over the proposition denoted by $\forall x. \varphi$ itself, too.

For the question of how to refer to antinomies in $\in_T$-logics consider for example the well-known antinomy of the Liar paradox given by the sentence "This sentence is false." Considering the Liar paradox to be true, it results in being false, and vice versa. The Liar paradox arises from the fact that it is both self-referential (given by "This sentence") and contains negation (given by "is false" in the sense of not being true). There are several ways to address the problem of antinomies in logic. Firstly, one can limit the formulation of referential sentences so that only consistent sentences can be formulated. It is obvious that the resulting logic is very limited in its ability to state referential sentences. Secondly, one can extend the notion of truth and falsity by introducing further truth values like for example the truth value unknown. By stating

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Despite its naming the class of $\in_\mu$-logics can be seen as a logic of the family of $\in_T$-logics, too.

See for example [5] for a good introduction into the field of antinomies.
the truth value of antinomies like the liar paradox as being unknown one can handle a certain set of antinomies, but nevertheless not all self-referential sentences. The proposed solution here is to introduce further truth values whereby an infinite hierarchy of partial truth predicates has to be defined (this approach is due to Kripke, see [22]).

In $\mathcal{ET}$-logics antinomies are being referred to by unsatisfiable equations. The Liar paradox proposition (that means the proposition denoted by the Liar paradox) is not a proposition in the $\mathcal{ET}$-logics as all of these logics are being proven to be free from antinomies. Nevertheless one can implicitly refer to the Liar proposition in form of the equation

$$x \equiv x:\text{false}.$$ 

Here $\equiv$ again states the propositional identity of the connected formulas, while $x:\text{false}$ denotes the proposition that the proposition assigned to the variable $x$ is false. This equation does not have a solution in $\mathcal{ET}$-logics as there is no assignment to $x$ which satisfies the equation. However, any conceivable solution to this equation would result in being equivalent to the Liar paradox proposition. That means that the Liar proposition (and other antinomies in the same way) can be referred to by unsolvable equations in $\mathcal{ET}$-logics without being actually available as a proposition in the semantics.

2 Propositions

In this section we want to take a closer look at the different understanding of the notion of propositions in the literature to form a basis for our understanding of the notion in the present work. The studies on propositions go back to the time of the Greek philosophy. When studying the conceptions of propositions in the literature, two aspects become clear: first, that there is a clear connection between propositions and logics, and second, that it is not possible to define the notion of propositions without considering the notion of judgments. With the different conceptions of propositions in the literature two questions arise which are interesting for the present work as well:

1. Are propositions syntactical or semantical entities?

2. Is there a clear distinction between a proposition and a judgment?

As we are mainly interested in the definition of a class of propositional logics in this work whose models can be interpreted as theories of propositions our view on the term of propositions is of course based on the view of logic. Thus in the view of the present work a proposition can be explicated as follows:

A proposition is the inherent sense of a formal expression.

A more general definition of propositions which is not based on the presence of a formal expression (but including the definition above) is given by Mahr in [27]:

That which is judged is a proposition. A proposition is what is true or false.

The two definitions show a very important aspect of the understanding of propositions in logics, namely that in the first definition propositions must be referencable by a
formal expression, while in the second definition a proposition may be independent from this referencability by any syntactical objects. Considering again the mentioned relationship between propositions and judgments, the first definition only seems to go without a notion of judgment. This is due to the fact that for interpreting a formal expression one always needs a certain concept of semantics. Thus the sense of a formal expression is not given just like that, but depends on the semantical surroundings. This idea corresponds to the conception that a judgment is always related to a certain subject who is judging. In the case of logics this subject can be seen as the currently considered model $\mathcal{M}$ under which a formal expression is being interpreted. In this understanding for example the validity of a formal expression $\phi$ in $\mathcal{M}$, written

$$\mathcal{M} \models \phi$$

forms the judgment that the proposition denoted by $\phi$ in the context of the model $\mathcal{M}$ is true in the context of the model $\mathcal{M}$.

In the year 1942 Carnap criticized (see [9]) that in the English language the term of proposition is used in different opponent ways: as a purely syntactical notion for a declarative sentence and as the semantical notion for the sense of a declarative sentence. Due to Carnap the second interpretation of propositions as semantical entities corresponds to the usage of the term in modern logic while (due to Robering, see [35]) the interpretation of propositions as syntactical entities goes back to the works of Aristotle. Although there are sporadic critics to the view of Carnap\footnote{Compare for example the remarks of Robering in [35] concerning the critics of Quine who rejects the existence of propositions as semantical entities.}, in today's understanding propositions are in fact meant to represent semantical entities. While in the view of Carnap propositions are referencable by declarative sentences, it is for example the understanding of Bolzano (see [8] or [7]) that there are propositions beyond those which can be referred to by syntactical entities.

While the notions of propositions and judgments were studied without the guidance of a formal semantical framework for a long time it was Frege who started to take a look at this conception form a more formal viewpoint (see [14] and compare the remarks of Mahr in [27]). A judgment in the view of Frege is not given as the plain affirmation or denial of a proposition, but as the explicit assertion of that affirmation or denial. In his Begriffsschrift Frege introduces certain formal notations for assertions which add a more formal understanding also to the notions of propositions and judgments. The Fregean understanding of judgments and propositions and the ideas of a formal system of deduction were picked up and elaborated by Martin-Löf in his intuitionistic type theory (see [30] and compare with [31]), which represents the modern understanding of the terms of propositions and judgments.

Another interesting and modern definition of the notion of propositions is given by Mahr in his model of conception (see [27]). In the understanding of his work one could argue that propositions are conceptions in his understanding of the term. Again these conceptions may exist without a referencable form.

According to the ideas of Martin-Löf the formulas of the $\in\mu$-logics as presented in this work denote propositions, but are not propositions themselves. Nevertheless in the semantics of $\in\mu$-logics the semantics laid to the operators of the formulas must be
sustained or reflected in the interpretation of formulas as propositions. This need for a reflection, however, does not purport the construction of propositions. As propositions do not necessarily have to be assignable in written form in the general case we cannot give rules for the construction of propositions and have to stick to the definition that propositions are semantical concepts representing the inherent sense or idea or state of affairs denoted by an entity, in the case of logics by a formula. However, while we do not want to make any restrictions to the definition of propositions in the general setting it is natural that we have to make some restricting assumptions when we want to explicitly reason about propositions in the semantics of a logic. These restrictions include for example that a proposition must be denotable by a formula which is an assumption not necessary in the general definition of propositions. Another assumption is for example that there is a clear connection between the connectives used in a formula and the proposition denoted by that formula. In the following we give a list of natural assumptions which are laid to our understanding of the term of proposition in the case of an explicit treatment of propositions in a logic.

Explication 2.1 (Assumptions on the notion of propositions in logics). An explicit formal treatment of propositions in a logic must fulfill the following assumptions:

1. Propositions can be referred to by formal expressions.
2. Propositions have an identity which allows for the comparison of propositions as being equal or not and which is not necessarily given as the equality of truth values.
3. Propositions can be parted into two worlds, a world of true propositions and a world of false propositions. The both worlds must be disjoint in any semantical context.
4. Propositions are closed under the means of expression of the logic. That means for example that propositions can be negated or placed into the context of a modal operator. The semantics of the operators of a formula can be carried over to the interpretation of the formula as a proposition.
5. The referencability of propositions by formal expressions is consistent in the sense that if two formal expressions $\phi_1$ and $\phi_2$ denote the same proposition, then the proposition denoted by a formal expression $\psi$ including $\phi_1$ as a sub-sentence must be preserved when replacing $\phi_1$ with $\phi_2$, and vice versa.
6. It is possible to universally quantify over the range of propositions.

While the first five points of the above list do fully reflect our understanding of the assumptions to be fulfilled by any logic which is explicitly treating propositions, the last point of the list is not really essential for such a treatment. However, from the viewpoint of the reconstruction of natural language semantics a quantification over propositions is an desirable property, as in natural language one often has to deal with assertions reasoning about the existence of propositions or about all available propositions. As we will see the above assumptions are fully reflected in the definition of the syntax and semantics of $\in\mu$-logics. Thus – if one accepts the above assumptions and restrictions
on the range of propositions to be valid – the class of $\varepsilon_\mu$-logics can be interpreted as an adequate theory of propositions (compare with Section 5).

It remains to answer the question why an explicit treatment of propositions in a logic is superior to the implicit meta-level understanding of the sense of a formula as present in traditional propositional logics. The answer to this question lies in the incapability of classical logics to compare the senses of formal expressions on the semantical level as the senses are only given by the truth or falsity of the formulas. As shown in the examples of the previous section this categorization is much too limited to reconstruct the semantics of natural language in logic, which is a topic of still increasing interest in computer science. A theory of an explicit interpretation of formal expressions as propositions furthermore allows for the definition of strong meta-level logics which in turn allow for the reasoning about underlying object-levels. The class of $\varepsilon_\mu$-logics as introduced in the following sections offers such a framework for the meta-level reasoning about arbitrary object-level logics by the interpretation of formulas as propositions.

3 Abstractions of logics and modal operators

As already mentioned in the introduction of this work the members of the class of $\varepsilon_\mu$-logics are characterized by the extension of underlying logics by a propositional layer together with the integration of additional means of expression for the extension of the meta-level reasoning capability of the logics. The field of universal representations of logics has become a well-studied field in logic over the last decades. For the purposes of the present work mainly two concepts for the abstract representation of logics are essential. In the following we introduce the concept of logics in abstract form according to Zeitz in [39] and the concept of integrations forms as representation forms for modal operators and their integration into arbitrary logics as defined by Bab in [1]. The concept of logics in abstract form gives a very easy-to-handle form of representation of a logic, but with that shows a very high degree of abstraction from the original syntactical and semantical concepts of a logic. Although the concept of logics in abstract form is suiting very well for the extension of logics in $\varepsilon_\mu$-logics the level of abstraction is too high for defining a general methodology for the integration of modal operators represented in an integration form into an arbitrary logic. To define such a general integration methodology we define another concept of an abstract representation of a logic, the logics in model-theoretic form as first defined by Bab and Mahr in [28].

The concept of logics in abstract form is defined as follows:

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10 This question is related to similar questions for example in the field of the combination and integration of logics (see for example the works of Gabbay on fibring in [16] or Finger and Gabbay in [13], the works on institutions by Meseguer in [32] and Goguen and Rosu in [17], as well as the works on heterogeneous specifications and the moving between logical systems by Tarlecki in [37, 38], Mossakowski in [33] and Diaconescu in [11]). Contrary to an equal combination of logics we make a clear distinction in this work between the meta-level $\varepsilon_\mu$-logic and the underlying object-level logic which is extended or integrated by the $\varepsilon_\mu$-logic.

11 For a good introduction into the field of abstract logics see for example [18].

12 For a more detailed discussion on logics in abstract form see for example [39, 1, 28].
**Definition 3.1** (Logics in abstract form). A logic in abstract form $\mathcal{L} = (\mathcal{L}, \mathcal{B})$ is given by a set of formulas $\mathcal{L}$ and a set $\mathcal{B} \subseteq \mathcal{P}(\mathcal{L})$ of subsets of $\mathcal{L}$ which is called the basis.

In this definition the elements of the basis of a logic in abstract form can be interpreted as the theories of the semantical concepts of the represented logic, although the concept is general enough to cover logics which do not have a notion of theories. In general any set of formulas $\mathcal{B} \in \mathcal{B}$ represents the maximal set of formulas which are true under a certain existing semantical concept of the logic.\(^{13}\) Although the level of abstraction is very high in the concept of logics in abstract form the definition allows for the representation of standard logical concepts like for example logical consequence or notions of tautology. We summarize the results of Zeitz in the following fact:

**Fact 3.2** (Logical concepts for logics in abstract form). For any logic in abstract form $\mathcal{L} = (\mathcal{L}, \mathcal{B})$ the following concepts can be derived (see [39]):

1. The logical consequence relation $\vdash_{\mathcal{B}}$ is defined for all sets of formulas $\Phi$ and all formulas $\phi$ by $\Phi \vdash_{\mathcal{B}} \phi$ iff $\Phi \subseteq B$ implies $\phi \in B$ for all $B \in \mathcal{B}$.
2. A formula $\phi$ is called consistent if there is at least one $B \in \mathcal{B}$ with $\phi \in B$.
3. A formula $\phi$ is called tautological if $\phi \in B$ for all $B \in \mathcal{B}$.

As mentioned before we will now define another form of an abstract representation of a logic, the concept of logics in model-theoretic form. While in the concept of logics in abstract form it is the notion of logical consequence which is in the center of the semantical considerations, it is the notion of models and the validity of formulas in these models which is central in the definition of logics in model-theoretic form.

**Definition 3.3** (Logics in model-theoretic form). A logic in model-theoretic form is given as a tuple $\mathcal{L} = (\mathcal{L}, \text{Mod}, \models)$ consisting of a set of formal expressions $\mathcal{L}$, a class of models $\text{Mod}$ and a validity relation $\models \subseteq \text{Mod} \times \mathcal{L}$ stating the validity of formal expressions in the models of the logic.

As for the logics in abstract form it is possible to derive standard logical concepts like for example logical consequence or notions of tautology in the framework of logics in model-theoretic form. We summarize the results in the following fact:

**Fact 3.4** (Logical concepts for logics in model-theoretic form). For any logic in model-theoretic form $\mathcal{L} = (\mathcal{L}, \text{Mod}, \models)$ the following derived concepts can be defined (see [28, Definition 2, Theorem 1]):

1. A formula $\phi$ resp. a set of formulas $\Phi$ is called valid in a model $\mathcal{M} \in \text{Mod}$ if $\mathcal{M} \models_{\text{Mod}} \phi$ resp. $\mathcal{M} \models_{\text{Mod}} \Phi$.
2. The logical consequence relation $\models_{\text{Mod}}$ is defined by $\Phi \models_{\text{Mod}} \phi$ iff for all $\mathcal{M} \in \text{Mod}$ it holds that $\mathcal{M} \models_{\text{Mod}} \Phi$ implies $\mathcal{M} \models_{\text{Mod}} \phi$.

\(^{13}\)To give an example, one could consider the representation of traditional propositional logic as a logic $\mathcal{L} = (\mathcal{L}, \mathcal{B})$ in abstract form. Here $\mathcal{L}$ is defined as the set of all formulas of the logic over a certain set of variables $P$, while $\mathcal{B}$ is defined as the set of theories $\text{Th}(\beta)$ for all assignments of variables to truth values $\beta : P \rightarrow \{T, F\}$. 

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3. A formula \( \varphi \) is called consistent if there is at least one \( M \in \text{Mod} \) such that \( \varphi \) is valid in \( M \).

4. A formula \( \varphi \) is called tautological if \( \varphi \) is valid in all \( M \in \text{Mod} \).

The two concepts of logics in abstract form and in model-theoretic form are equivalent in the sense that any logic represented in the one abstraction form can be transformed into the other abstraction form and vice versa such that the logical concepts of the logics coincide. We state the results of Bab and Mahr in [28] in the following Fact:

Fact 3.5 (Transformation between abstraction forms). The following results on the transformation between logics in abstract form and logics in model-theoretic form do hold:

1. Transformation into abstract form, [28, Theorem 3]: Every logic in model-theoretic form \( L \) can be conservatively transformed into a logic in abstract form \( L' \) such that:
   - The consequence relations of \( L \) and \( L' \) are equal.
   - Any formula \( \varphi \) is consistent in \( L \) iff \( \varphi \) is consistent in \( L' \).
   - Any formula \( \varphi \) is tautological in \( L \) iff \( \varphi \) is tautological in \( L' \).

2. Transformation into model-theoretic form, [28, Remarks on page 7]: Every logic in abstract form \( L \) can be conservatively transformed into a logic in model-theoretic form \( L' \) such that:
   - The consequence relations of \( L \) and \( L' \) are equal.
   - Any formula \( \varphi \) is consistent in \( L \) iff \( \varphi \) is consistent in \( L' \).
   - Any formula \( \varphi \) is tautological in \( L \) iff \( \varphi \) is tautological in \( L' \).

The idea of the proofs of these relationships can be summarized as follows: the transformation from a logic in abstract form into a logic in model-theoretic form can be achieved by interpreting the theories of the logic as represented in the basis as models in the resulting logic. A formulas then is valid in such a model if and only if it is a member of the theory. The transformation from the model-theoretic form into the abstract form involves the construction of the theories of the models which will form the elements of the basis of the logic in abstract form. Here it must be mentioned that a transformation and re-transformation of a logic in model-theoretic form does necessarily not result in exactly the same logic due to the abstractions taking place, but in a logic which is isomorphic in the truth and falsity of formulas.

As stated before every \( \in \mu \)-logic allows for the extension of an underlying logic in abstract form. Here extension is meant in the sense of the following definition as preserving syntactical and semantical properties:

Definition 3.6 (Extension of logics). Let \( L_1, L_2 \) be arbitrary logics with logical consequence relations \( \vdash_{L_1} \) and \( \vdash_{L_2} \). \( L_2 \) is said to extend the logic \( L_1 \) if the following conditions are satisfied:
Every formula of $L_1$ is a formula of $L_2$.

For all sets of formulas $\Phi$ and all formulas $\varphi$ of $L_1$ it holds $\Phi \vdash_{L_1} \varphi$ if and only if $\Phi \vdash_{L_2} \varphi$.

As we will see in the next section every $\mathcal{E}_\mu$-logic contains a common basic set of operators which can be extended by the integration of arbitrary truth functional operators and certain modal operators over a possible worlds semantics. Here the basic set of propositional operators includes classical negation and implication as a basis of classical connectives, an operator for the propositional identity of formal expressions, and a universal quantifier for the quantification over all available propositions. In the following we introduce a representation form for truth functional operators and modalities – the so-called integration form of operators – and show how arbitrary logics can be enriched by the represented operators. The proposed approach forms the basis for the integration of operators into $\mathcal{E}_\mu$-logics in the next section. An integration form of operators is defined as follows:

**Definition 3.7** (Integration form of operators). An integration form of a set of operators is given as $M = (\mathcal{C}, s, \mathbb{K}, \mathbb{F})$ where $\mathcal{C}$ is a set of operator symbols, $s : \mathcal{C} \to \mathbb{N}$ is a function which states the arity of the operators, $\mathbb{K}$ is a class of Kripke structures$^{14}$ which forms the basis for the semantical interpretation of the operators, and $\mathbb{F}$ contains for every $c \in \mathcal{C}$ a semantical interpretation function $f_c$ of one of the following forms:

(a) $f_c : \{T, F\}^{s(c)} \to \{T, F\}$.

(b) $f_c : N^{s(c)}(\mathbb{K}) \to \{T, F\}$ where

$N^{s(c)}(\mathbb{K}) := \{(W, R), w, V) \mid (W, R) \in \mathbb{K}, w \in W, V \in \mathbb{V}^{s(c)}(W)\}$ and

$\mathbb{V}^{s(c)}(W) := \{V \mid V^{s(c)} : W \to \{T, F\}^{s(c)}\}$.

Operators with functions of the first type are called local operators, whereas operators with functions of the second type are called modal operators.

At first sight the above definition of integration forms of modal operators seems to be rather complicated while the representation of local operators corresponds to the typical definition of truth functions for classical connectives. However, especially in the case of modal operators it is necessary to reproduce the semantics of the operator in the general case by their definitions in a possible worlds model, which is accomplished here by a much wider definition of the term of truth functions.

This more complex version of truth functions is necessary, because due to Kripke (see for example [20, 21]) the semantics of modal operators is often determined by the choice of possible world structures available for the semantics. The interpretations of the modality necessary in the different logical systems (due to Lewis, see for example [23, 24]) of classical modal logic for example is directly related to structural properties of the available possible world structures. Here for example the system $T$ corresponds to the use of Kripke structures having a reflexive accessibility relation, while the system

---

$^{14}$ A Kripke structure is defined in the usual way as $\mathcal{K} = (W, R)$, where $W$ is set of worlds and $R \subseteq W \times W$ is an accessibility relation on the worlds of $W$. For a more detailed introduction into the field of modal logics and possible worlds (or Kripke) semantics see for example [19, 34, 10].
$S4$ corresponds to Kripke structures which are reflexive and transitive. In the proposed concept of integration forms we directly represent these circumstances by adding the class of Kripke structures to the integration form, which forms the basis for the interpretation of the operators. The corresponding semantical interpretation functions then state the behaviour of the operators in all Kripke structures and from the viewpoint of every existing world. In this definition we can obviously only represent operators the semantics of which is based on the truth values of their argument formulas in the different worlds of the considered Kripke structure. Fortunately most of the standard modalities do fit into this definition.

Consider for example the modality necessary of the logical system $S4$ of classical modal logic. An integration form representing this modal operator would have a symbol representing the operator (for example $\Box \in C$ with arity $s(\Box) = 1$), the class of reflexive and transitive Kripke structures as $K$, and a function $f_\Box$ defined in the following way:

$$f_\Box((W,R),w,V) := T$$

$$\Leftrightarrow$$

for all $w' \in W$ with $(w,w') \in R$ it holds $(w',T) \in V$

The concept of integration forms of modal operators is basic for the extension of the means of expression of single $\mu$-logics as introduced in the following section. However, the proposed concept of integration forms and the integration of the represented operators can be used as a general basis for the integration of truth-functional operators into arbitrary logics. We want to sketch this approach in the following considerations and re-use the concept in the next section for the definition of $\mu$-logics.

The central task in any uniform representation and integration of operators into arbitrary logics is the task of guaranteeing that the representations and integrations are fully preserving the semantics of the operators, such that the integrated operators behave in exactly the same way as intended. Due to the fact that we only represent operators which are in a sense truth-functional in our integration forms and by fully representing the class of available Kripke structures we can guarantee that the represented operators are fully preserving their semantics in the integration process.

In the following we show how a logic in model-theoretic form can be enriched by a possible worlds semantics which must be the first step before a semantic preserving integration of modal operators can take place. The main idea is to define the semantics relative to a given Kripke structure where any world is laid to a model of the given logic in model-theoretic form. Here the Kripke structure must be given by the integration form of the operators to be integrated to guarantee a semantic preserving integration.

We define:

**Definition 3.8** (Extension by a possible worlds semantics). Let $\mathcal{L} = (L, \text{Mod}, \models_{\text{Mod}})$ be an arbitrary logic in model-theoretic form and $K$ an arbitrary class of Kripke-structures. The extension of $\mathcal{L}$ by the possible worlds structures of $K$ is defined as a logic in model-theoretic form $\mathcal{L}(K) = (L, \text{Mod}(K), \models_{\text{Mod}(K)})$ where the following holds:

15 A much more detailed study of the following concepts can be found in [1].
1. For any Kripke-structure $\mathcal{K} = (W, R) \in K$ we call a function $h : W \rightarrow Mod$ an assignment of $\mathcal{K}$. Further let

$$\text{Mod}_\mathcal{K} := \{ (\mathcal{K}, w, h) \mid w \in W \text{ and } h : W \rightarrow Mod \text{ is an assignment} \}$$

be the class of all models respective to $\mathcal{K}$. The class $\text{Mod}(\mathcal{K})$ of the models of the logic $L(\mathcal{K})$ is then defined by:

$$\text{Mod}(\mathcal{K}) := \bigcup_{\mathcal{K} \in K} \text{Mod}_\mathcal{K}.$$

2. The validity relation $\models_{\text{Mod}(\mathcal{K})} \subseteq \text{Mod}(\mathcal{K}) \times L$ of the logic $L(\mathcal{K})$ is defined for all models $(\mathcal{K}, w, h) \in \text{Mod}(\mathcal{K})$ and all formal expressions $\varphi \in L$ by:

$$(\mathcal{K}, w, h) \models_{\text{Mod}(\mathcal{K})} \varphi \iff h(w) \models \varphi.$$

The proposed integration results in a logic where each world of each model represents a model of the original logic. As there are no new operators yet there cannot be any reasoning over the circumstances in other worlds from the viewpoint of an arbitrary given world. This will be possible after integrating modal operators represented in an integration form which will be our next step.

The idea of the syntactical integration is quite easy. It is based on the representation of the set of formal expressions in an inductively defined way:

**Definition 3.9 (Inductively defined sets of formulas).** A set of formulas $L$ is said to be inductively defined if the following holds:

1. $L := L(A, \langle C, s \rangle)$ where $A$ is a set of atomic formulas, $C$ is a set of operators, and $s : C \rightarrow \mathbb{N}$ is a function mapping each operator to its arity.

2. $L$ is the smallest set of formulas with the following properties:
   - $A \subseteq L$
   - For every $c \in C$ and every $\varphi_1, \ldots, \varphi_{s(c)} \in L$ it holds that:
     $$c(\varphi_1, \ldots, \varphi_{s(c)}) \in L.$$

Without loss of generality we assume the set of operators of a given logic $L$ and the set of operators to be integrated into $L$ to be disjoint. We define the syntactical integration as follows:

**Definition 3.10 (Syntactical Integration).** Let $L := L(A, \langle C', s' \rangle)$ be a set of inductively defined formal expressions and let $\mathcal{M} = (\langle C, s \rangle, \mathcal{K}, F)$ be an arbitrary integration form. The syntactical integration of the operators of $\mathcal{M}$ into $L$ is defined as the smallest set of formal expressions $L(\mathcal{M})$ which satisfies the following conditions:

(a) $A \subseteq L(\mathcal{M})$. 

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(b) For all operators \( c \in C' \) it holds:
\[
\phi_1, \ldots, \phi_{s(c)} \in L(M) \Rightarrow c(\phi_1, \ldots, \phi_{s(c)}) \in L(M).
\]

(c) For all operators \( c \in C \) it holds:
\[
\phi_1, \ldots, \phi_{s(c)} \in L(M) \Rightarrow c(\phi_1, \ldots, \phi_{s(c)}) \in L(M).
\]

In the following we assume \( \mathcal{L}(M) \) to be the logic which results from the extension of a logic \( \mathcal{L} \) by the possible worlds structures given by the integration form \( M \) and the syntactical integration of the represented operators. It remains to show how the validity of the new formulas in the models of the new logic can be defined. Therefore we must define \( \models^{\text{Mod}(k)} \) to be an adequate extension of the validity relation \( \models^{\text{Mod}(k)} \) of the original logic. Here we must ensure that the validity of the previously existing formulas is being preserved in the corresponding models and that the semantical integration of the new operators is semantic preserving. We define the semantical integration as follows:

**Definition 3.11 (Semantical integration).** In the following let \( M = (C, s, \mathcal{K}, \mathcal{F}) \) be an arbitrary integration form and \( \mathcal{L}(k) = (L, \text{Mod}(k), \models^{\text{Mod}(k)}) \) be the resulting logic of the extension of \( \mathcal{L} \) by a possible worlds semantics and the syntactical integration of the operators of \( M \). The semantical integration of the operators of \( M \) is defined by the following recursively defined extension of the validity relation \( \models^{\text{Mod}(k)} \) into a resulting relation \( \models^{\text{Mod}(k)} \subseteq \text{Mod}(k) \times L(M) \). First let \( \models_{\text{Mod}(k)}^{*} \) be equal to \( \models^{\text{Mod}(k)} \). The we define for all operators \( c \in C \cup C' \):

- If \( c \in C' \) the validity of formulas having \( c \) as the outer operator has to be defined according to the semantics of the logic \( \mathcal{L} \) which has to be extended in an adequate way to formulas containing the new operators as arguments.\(^{16}\)

- If \( c \in C \) is a local operator with truth-function \( f_c \) we define for all \( \phi_1, \ldots, \phi_{s(c)} \in L(k) \) and all models \( (\mathcal{K}, w, h) \in \text{Mod}(k) \)
\[
(\mathcal{K}, w, h) \models^{*}_{\text{Mod}(k)} c(\phi_1, \ldots, \phi_{s(c)}) \iff f_c(v_1, \ldots, v_{s(c)}) = T
\]
where the truth values \( v_i \) are defined for all \( i \in \{1, \ldots, s(c)\} \) by
\[
v_i := \begin{cases} 
  T, & \text{if } (\mathcal{K}, w, h) \models^{*}_{\text{Mod}(k)} \phi_i \\
  F, & \text{if } (\mathcal{K}, w, h) \not\models^{*}_{\text{Mod}(k)} \phi_i
\end{cases}
\]

- If \( c \in C \) is a modality with truth-function \( f_c \) we define
\[
(\mathcal{K}, w, h) \models_{\text{Mod}(k)}^{*} c(\phi_1, \ldots, \phi_{s(c)}) \iff f_c(\mathcal{K}, w, V^{s(c)}) = T,
\]

\(^{16}\)It should be noted here that this step cannot be made explicit without knowing the semantics of the operators of the logic \( \mathcal{L} \). If the operators are truth-functional the proposed extension can be made very easily. In any other case the semantics of the operators has to be extended by hand. To keep the given procedure as general as possible we do not want to restrict the original set of operators of \( \mathcal{L} \) to a more easy choice like for example truth-functional operators.
where the truth configuration $V^{s(c)}$ is defined in the following way for all $w' \in W$:

$$V^{s(c)}(w') := (v_1, \ldots, v_{s(c)}) \text{ mit } v_i := \begin{cases} T, & \text{if } (\mathcal{N}, w', h) \models^s_{\text{Mod}(K)} \varphi_i \\ F, & \text{if } (\mathcal{N}, w', h) \not\models^s_{\text{Mod}(K)} \varphi_i \end{cases}$$

The proposed semantical integration is recursive in the way that the validity of a formula $\varphi$ having one of the integrated operators as the outermost operator is depending on the truth values of the argument formulas which is represented in the truth configurations. These argument formulas however can include integrated operators, too, so that the validity of these formulas again must be determined by the procedure described above.

We have defined a procedure which can be used as a basis for an easy integration of truth-functional local or modal operators into two-valued logics. Nevertheless the procedure needs for an adjustment to the semantics present in the logic in which the operators shall be integrated into to achieve a semantically adequate integration with the other operators of the logic. We will use the proposed procedure for the integration of operators into $\in_{\mu}$-logics in the next section.

4 $\in_{\mu}$-logics

In this section we will now introduce the class of $\in_{\mu}$-logics. Based on the conceptions of logics in abstract form and the integration forms of operators as defined in the previous section any $\in_{\mu}$-logic depends on two parameters:

1. A certain given logic in abstract form $\mathcal{L}$ which acts as the underlying object-level logic which is to be extended by a propositional layer in the $\in_{\mu}$-logic. Here the formulas of $\mathcal{L}$ become constants in the resulting $\in_{\mu}$-logic.

2. A set of propositional operators given as an integration form $\mathcal{M}$ which are integrated as additional propositional means of expression into the $\in_{\mu}$-logic.

The $\in_{\mu}$-logic over $\mathcal{L}$ and $\mathcal{M}$ is denoted by $\in_{\mu}(\mathcal{L}, \mathcal{M})$. The syntax of an $\in_{\mu}$-logic $\in_{\mu}(\mathcal{L}, \mathcal{M})$ is defined as:

**Definition 4.1 (Syntax of an $\in_{\mu}(\mathcal{L}, \mathcal{M})$-logic).** Let $\mathcal{L} = (L, \mathcal{R})$ be an arbitrary logic in abstract form, $\mathcal{M} = ((\mathcal{C}, s), \mathcal{K}, \mathcal{F})$ an integration form of a set of operators, and $X = \{x_i \mid i \in \mathbb{N}\}$ a well-ordered set of variables. The set of formulas $L_{\mu}$ of the $\in_{\mu}(\mathcal{L}, \mathcal{M})$-logic is defined as follows:

1. $X \subseteq L_{\mu}$ and $L \subseteq L_{\mu}$.
2. $\varphi \in L_{\mu}$ implies $\neg \varphi \in L_{\mu}$.
3. $\varphi, \psi \in L_{\mu}$ implies $(\varphi \rightarrow \psi), (\varphi \equiv \psi) \in L_{\mu}$.
4. $\varphi \in L_{\mu}$ and $x \in X$ implies $(\forall x. \varphi) \in L_{\mu}$.
5. For all $c \in \mathcal{C}$: $\varphi_1, \ldots, \varphi_{s(c)} \in L_{\mu}$ implies $c(\varphi_1, \ldots, \varphi_{s(c)}) \in L_{\mu}$.
In this definition syntactical integration takes place on two different levels. On the one hand there is an integration of the formulas of the underlying logic in abstract form as atomic entities in the $\mathcal{L}_\mu$-logic. Notably there is no integration of the operators of the object-level logic as general operators in the meta-level logic. On the other hand there is an integration of the operators of the integration form as means of expression which are on the same level as the basic operators of the $\mathcal{L}_\mu$-logic.  

The semantics of $\mathcal{L}_\mu$-logics is defined in terms of $\mathcal{L}_\mu$-structures. Any $\mathcal{L}_\mu$-structure includes a sense function $\Gamma$ which interprets formulas as propositions depending on an indicated world and an assignment of variables to propositions. Any $\mathcal{L}_\mu$-structure offers a set of available propositions which function as the range for the sense function. Propositions can be either true or false depending on the considered world. The interpretation of formulas as propositions is not arbitrary, but must be compliant with the semantics of the included operators and the logical laws. We define the concept of $\mathcal{L}_\mu$-structures as follows:

**Definition 4.2 ($\mathcal{L}_\mu$-structure).** Let $\mathcal{M} = (\mathcal{L}, \mathcal{M})$ be the $\mathcal{L}_\mu$-logic over $\mathcal{L} = (L, \mathcal{B})$ and $\mathcal{M} = (\mathcal{C}, s), \mathcal{K}, \mathcal{F})$. An $\mathcal{L}_\mu(\mathcal{L}, \mathcal{M})$-structure is defined as $\mathcal{N} = (\mathcal{K}, M, T, \Gamma)$ where:

1. $\mathcal{K} = (W, R) \in \mathcal{K}$ is a Kripke structure.
2. $M$ is a non-empty set of propositions and $T = (T_w)_{w \in W}$ is a family of non-empty sets $T_w \subseteq M$, which state the true propositions in the worlds of $W$.
3. $\Gamma : L \times W \times [X \rightarrow M] \rightarrow M$ is the sense function which interprets formulas as propositions, where the following conditions hold:

   (a) Truth properties:
   
   For every $w \in W$ and every propositional assignment $\beta : X \rightarrow M$ it holds:
   
   1. $\Gamma(\neg \varphi, w, \beta) \in T_w$ if and only if $\Gamma(\varphi, w, \beta) \notin T_w$.
   2. $\Gamma(\varphi \rightarrow \psi, w, \beta) \in T_w$ if and only if $\Gamma(\varphi, w, \beta) \notin T_w$ or $\Gamma(\psi, w, \beta) \in T_w$.
   3. $\Gamma(\varphi \equiv \psi, w, \beta) \in T_w$ if and only if $\Gamma(\varphi, w, \beta) = \Gamma(\psi, w, \beta)$.
   4. $\Gamma(\forall x. \varphi, w, \beta) \in T_w$ if and only if for all $m \in M$ such that $\Gamma(\varphi, w, \beta[x/m]) \in T_w$.

   Here $\beta[x/m](y) := \begin{cases} m, & \text{if } y = x \\ \beta(y), & \text{if } y \neq x \end{cases}$

   (b) Contextual properties:

   For every $w \in W$ and every $\beta : X \rightarrow M$ it holds:
   
   i. $\Gamma(x, w, \beta) = \beta(x)$ for every $x \in X$.
   ii. For every $\beta_1, \beta_2 : X \rightarrow M$ with $\beta_1 \upharpoonright \{ \text{Free}(\varphi) \} = \beta_2 \upharpoonright \{ \text{Free}(\varphi) \}$ it holds that $\Gamma(\varphi, w, \beta_1) = \Gamma(\varphi, w, \beta_2)$.
   iii. For every $\varphi, \psi$ that are $\alpha$-congruent it holds that $\Gamma(\varphi, w, \beta) = \Gamma(\psi, w, \beta)$ in all worlds $w \in W$ and for every $\beta : X \rightarrow M$.
   iv. For the negation, the implication and the integrated local operators $\ell \in \mathcal{C}$ there are functions

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\[ ^{17} \text{Here Free}(\varphi) \] denotes the set of free variables in $\varphi$ in the usual way.

\[ ^{18} \text{$\alpha$-congruency is defined in the usual way, i.e. two formal expressions } \phi \text{ and } \psi \text{ are } \alpha \text{-congruent if they only differ in the names of their bound variables.} \]
\[ [-] : W \times M \rightarrow M, \]
\[ [\rightarrow] : W \times M \times M \rightarrow M \text{ and } \]
\[ [\ell] : W \times M^{(\ell)} \rightarrow M \]

such that for all \( \varphi, \varphi_1, \varphi_2, \cdots \in L_\mu \), all \( w \in W \) and all \( \beta : X \rightarrow M \) the following holds:

\[
\Gamma(\neg \varphi, w, \beta) = [-](w, \Gamma(w, \beta))
\]
\[
\Gamma(\varphi_1 \rightarrow \varphi_2, w, \beta) = [\rightarrow](w, \Gamma(\varphi_1, w, \beta), \Gamma(\varphi_2, w, \beta))
\]
\[
\Gamma(\ell(\varphi_1, \ldots, \varphi_{s(\ell)}), w, \beta) = [\ell](w, \Gamma(\varphi_1, w, \beta), \ldots, \Gamma(\varphi_{s(\ell)}, w, \beta))
\]

(c) Integration properties:

i. For all local constructors \( \ell \in C \), all formulas \( \varphi_1, \ldots, \varphi_{s(\ell)} \), all \( w \in W \) and all \( \beta : X \rightarrow M \) it holds that:

\[
\Gamma(\ell(\varphi_1, \ldots, \varphi_{s(\ell)}), w, \beta) \in T_w \iff f_{\ell}(v_1, \ldots, v_{s(\ell)}) = T,
\]
where for all \( i \):

\[
v_i := \begin{cases} 
T, & \text{if } \Gamma(\varphi_i, w, \beta) \in T_w \\
F, & \text{if } \Gamma(\varphi_i, w, \beta) / \in T_w 
\end{cases}
\]

ii. For all modal constructors \( \Box \in C \), all formulas \( \varphi_1, \ldots, \varphi_{s(\Box)} \), all \( w \in W \) and all \( \beta : X \rightarrow M \) it holds that:

\[
\Gamma(\Box(\varphi_1, \ldots, \varphi_{s(\Box)}), w, \beta) \in T_w \iff f_{\Box}(X, w, V) = T,
\]
where for all \( w' \in W \) the function \( V \) is defined by

\[
V(w') := (v_1, \ldots, v_{s(\Box)}),
\]

where for all \( i \):

\[
v_i := \begin{cases} 
T, & \text{if } \Gamma(\varphi_i, w', \beta) \in T_{w'} \\
F, & \text{if } \Gamma(\varphi_i, w', \beta) / \in T_{w'}
\end{cases}
\]

The above definitions of the syntax and semantics of \( C_\mu \)-logics show that the interpretation of formal expressions as propositions is not arbitrary, but depends on the semantics of the operators, a consistent handling of the truth values, and the compliance with standard logical laws. Furthermore one has to answer the question if the truth values of propositions are the same for all worlds or if they can vary depending on the currently considered world. In our understanding of propositions this question can be answered easily. As a proposition is the entity representing the state of affairs or the inherent sense denoted by a formal expression a proposition can be true in the one world and false in the other. If we consider for example the state of affairs represented by the sentence

\[ \text{It is } 12:00 \text{ o’clock.} \]

then it is obvious that the proposition denoted by this sentence is true or false depending on the current time and with this on the currently considered world.
Another important question is if the interpretation of a formula as a proposition is the same in all worlds or can vary in different worlds. The answer to this question seems to be much more complicated. However, in this paper we are of the opinion that the proposition denoted by a formula is not fix, but depending on the considered world. This is due to the idea that in natural language the sense of a sentence often depends on the currently prevalent surroundings. This idea is a natural idea in logics, too. Consider for example again the formula
\[ \chi = \text{add}(x, \text{one}) = y \]
of first-order predicate logic as mentioned before. The inherent proposition denoted by this formula (that means the proposition beyond its truth value) depends on the logical structure in which the formula is going to be interpreted. As \( \epsilon_\mu \)-logics allow for the extension of arbitrary logics like for example first-order predicate logic and the usage of a different logical structures of predicate logic in any of the worlds of the Kripke structure of the \( \epsilon_\mu \)-model it is necessary to allow for the interpretation of formulas as different propositions in different worlds.

The class of \( \epsilon_\mu \)-logics allows for the definition of standard logical concepts like a notion of validity and a consequence relation. For the consequence relation we can show the validness of a certain deduction property.

**Definition 4.3** (Validity and consequence relation). Let \( \epsilon_\mu (\mathscr{L}, \mathcal{M}) \) be an arbitrary \( \epsilon_\mu \)-logic. Then the following concepts can be defined:

- The validity of formulas \( \varphi \) depending on considered world \( w \) under an assignment \( \beta \) in a model \( \mathcal{M} \) is defined by:
  \[ (\mathcal{M}, w, \beta) \models_\mu \varphi :\Leftrightarrow \Gamma(\varphi, w, \beta) \in T_w. \]

- The validity of sets of formulas is defined by:
  \[ (\mathcal{M}, w, \beta) \models_\mu \Phi :\Leftrightarrow (\mathcal{M}, w, \beta) \models \varphi \text{ for all } \varphi \in \Phi. \]

- A tuple \((\mathcal{M}, w)\) is called a model of a formula \( \varphi \) resp. a set of formulas \( \Phi \) – written \((\mathcal{M}, w) \models_\mu \varphi \) resp. \((\mathcal{M}, w) \models_\mu \Phi \) – if for all assignments \( \beta \) it holds that \((\mathcal{M}, w, \beta) \models \varphi \) resp. \((\mathcal{M}, w, \beta) \models \Phi \).

- A consequence relation can be defined as follows: For all sets of formulas \( \Phi \) and all formulas \( \varphi \) it holds that \( \Phi \models_\mu \varphi \) iff for all \( \epsilon_\mu (\mathscr{L}, \mathcal{M}) \)-structures \( \mathcal{M} \) and all included worlds \( w \) the following condition holds:
  \[ (\mathcal{M}, w) \models_\mu \Phi \Rightarrow (\mathcal{M}, w) \models_\mu \varphi \]

**Fact 4.4** (Deduction property). Let \( \epsilon_\mu (\mathscr{L}, \mathcal{M}) \) be an arbitrary \( \epsilon_\mu \)-logic. For all sets of formulas \( \Phi \) and all formulas \( \varphi \) and \( \psi \) it holds (see [1, Theorem 5.22]):

1. If \( \Phi \models_\mu \varphi \rightarrow \psi \) then it holds that \( \Phi \cup \{ \varphi \} \models_\mu \psi. \)
2. It is $\Phi \cup \{\varphi\} \models_{M} \psi$ if $\Phi \models_{M} \varphi^\forall \rightarrow \psi$. Here $\varphi^\forall$ is defined as the universal closure of $\varphi$: If $\text{Free}(\varphi) = \{x_1, \ldots, x_n\}$ then it is $\varphi^\forall := \forall x_1, \ldots, x_n, \varphi$.

3. It is $\Phi \models_{M} \varphi \iff \Phi \cup \{\neg \varphi^\forall\}$ does not have a model in $\mathcal{M}$.

As the definition of the semantics of $\mu$-logics is rather complex it is not easy to see that $\mu$-logics are free from inconsistencies. In order to show the consistency of the concept Bab proved the existence of certain extensional and intensional models of $\mu$-logics. We define the notion of extensional and maximal intensional $\mu$-structures and summarize the results of Bab in the following fact:

**Definition 4.5** (Extensional and maximal intensional $\mu$-structures). In the following let $\mathcal{M} = (M, T, \mathcal{X}, \Gamma)$ be an arbitrary $\mu$-structure with $\mathcal{X} = (W, R)$.

- $\mathcal{M}$ is called to be extensional if the set of available propositions $M$ has exactly two elements and if $T_{w_1} = T_{w_2}$ for all worlds $w_1, w_2 \in W$.
- $\mathcal{M}$ is called to be maximal intensional if for all sentences $\varphi, \psi \in \text{Sent}_{w}$ and all worlds $w \in W$ it is the case that if $\varphi \neq_{w} \psi$ it follows that $(\mathcal{M}, w) \not\models_{\mu} (\varphi \equiv \psi)$.

**Fact 4.6** (Existence of extensional and maximal intensional models). Let $\mu \in \mathcal{M}$ be an arbitrary $\mu$-logic with $\mathcal{L} = (L, \mathcal{B})$ and $\mathcal{M} = ((\mathcal{C}, \mathcal{S}), \mathcal{K}, \mathcal{F})$.

1. Existence of extensional models\textsuperscript{20}, [1, Theorem 6.3]: For all possible worlds structures $\mathcal{X} = (W, R) \in \mathcal{K}$ and all $(B_w)_{w \in W}$ with $B_w \in \mathcal{B}$ for all $w \in W$ there exists an extensional $\mu$-structure $\mathcal{M} = (M, T, \mathcal{X}, \Gamma)$ such that for all $w \in W$ it holds: $\{a \in L \mid \Gamma(a, w, \beta) \in T_w \text{ for all } \beta : X \rightarrow M\} = B_w$.

2. Existence of maximal intensional models, [1, Theorem 6.5]: For all possible worlds structures $\mathcal{X} = (W, R) \in \mathcal{K}$ and all $(B_w)_{w \in W}$ with $B_w \in \mathcal{B}$ for all $w \in W$ there exists a maximal intensional $\mu$-structure $\mathcal{M} = (M, T, \mathcal{X}, \Gamma)$ such that for all $w \in W$ it holds: $\{a \in L \mid \Gamma(a, w, \beta) \in T_w \text{ for all } \beta : X \rightarrow M\} = B_w$.

The existence of such models shows the consistency of the class of $\mu$-logics, however, for practical applications of these logics the theory of models does have to be studied further. In order to accomplish this Bab and Wieczorek showed how to create consistent (intensional\textsuperscript{21}) models of classical $\tau$-logic from a given relation of propositional equivalent formal expressions (see [4]). As stated in the outlook of [4] it is possible that the idea of the result can be adopted to other $\tau$-logics including the $\mu$-logics in the same way. As a result $\mu$-logics form a class of logics with special intensional models.

\textsuperscript{19}Here $\text{Sent}_{w}$ denotes the set of sentences, that means formulas which do not have free variables.

\textsuperscript{20}Any extensional model offers two propositions for being true and false. It should be noted here that there is still a clear distinction between the proposition denoted by a formula and the truth value of that formula (respectively the truth value of the proposition). Thus the construction of extensional models is not contrary to the understanding of Frege that the truth values themselves cannot be interpreted as propositions (compare with remarks in Section 2).

\textsuperscript{21}Here intensional refers to any model in which the interpretation of formulas as propositions is not limited to two propositions for being true and false.
As already stated in the introduction of this work the class of $\varepsilon_\mu$-logics covers the parameterized $\varepsilon_T$-logic by Zeitz and thus the classical $\varepsilon_T$-logic by Sträter. This is achieved by the integration of predicates for true and false over a special Kripke structure into the $\varepsilon_\mu$-logic as represented in the following integration form:

**Definition 4.7** (Integration form for true and false). Let $\mathcal{M}_{\varepsilon_T} := ((C,s), \mathcal{K}_{local}, \mathcal{F})$ with $\mathcal{K}_{local} := (\{w\}, \emptyset)$ be defined as follows:

- $C := \{\text{true}, \text{false}\}$ with $s(\text{true}) = s(\text{false}) = 1$.
- $F := \{f_{\text{true}}, f_{\text{false}}\}$ with $f_{\text{true}}, f_{\text{false}} : \{T,F\} \rightarrow \{T,F\}$ and $f_{\text{true}}(T) := T, f_{\text{true}}(F) := F$ and $f_{\text{false}}(T) := F, f_{\text{false}}(F) := T$.

We state the result of [1] in the following fact:

**Fact 4.8** (Covering parameterized $\varepsilon_T$-logic). Let $\varepsilon_T(L)$ be the parameterized $\varepsilon_T$-logic over an arbitrary logic in abstract form $L$ in the sense of Zeitz (see [39]) and let $\varepsilon_\mu(L, \mathcal{M}_{\varepsilon_T})$ be the corresponding $\varepsilon_\mu$-logic over $L$ and $\mathcal{M}_{\varepsilon_T}$.

1. [1, Theorem 6.10]: For all $\varepsilon_T(L)$-structures $\mathcal{M}_{\varepsilon_T}$ there exists an $\varepsilon_\mu(L, \mathcal{M}_{\varepsilon_T})$-structure $\mathcal{M}_{\varepsilon_\mu}$ such that for all formulas $\varphi \in L_\mu$ and all assignments $\beta : X \rightarrow M$ it holds that:

   $$(\mathcal{M}_{\varepsilon_T}, \beta) \models_{\varepsilon_T} \varphi \Leftrightarrow (\mathcal{M}_{\varepsilon_\mu}, w, \beta) \models_{\varepsilon_\mu} \varphi$$

2. [1, Theorem 6.10]: For all $\varepsilon_\mu(L, \mathcal{M}_{\varepsilon_T})$-structures $\mathcal{M}_{\varepsilon_\mu}$ fulfilling the substitution property stated by Zeitz\(^{22}\) there exists an $\varepsilon_T(L)$-structure $\mathcal{M}_{\varepsilon_T}$ such that for all formulas $\varphi \in L_\mu$ and all assignments $\beta : X \rightarrow M$ it holds that:

   $$(\mathcal{M}_{\varepsilon_\mu}, w, \beta) \models_{\varepsilon_\mu} \varphi \Leftrightarrow (\mathcal{M}_{\varepsilon_T}, \beta) \models_{\varepsilon_T} \varphi$$

In the following section we will now give a new interpretation of the class of $\varepsilon_\mu$-logic and its models as a theory of propositions which can be explicitly reasoned about in logics as discussed in Section 2.

## 5 $\varepsilon_\mu$-logics as a theory of propositions

As we have discussed in Section 2 our understanding of the notion of propositions in the general case is that a proposition is the inherent sense or idea of a certain entity. In this understanding propositions are semantical entities and it is not necessary that there is a dependency between propositions and certain syntactical concepts. In particular that means that in our general understanding of propositions there must be no referencability of propositions by written expressions. However, whenever one wants to explicitly reason about propositions in the setting of a logic, various natural assumptions have to be made, including the assumption that it must be possible to refer to propositions over the formulas of the logic. Hence in the field of propositional logics

\(^{22}\)In his parameterized $\varepsilon_T$-logic Zeitz defines a certain substitution property which allows for exchanging propositional equal parts in a formula, thereby preserving its denoted sense.
a proposition can be defined as the sense of a formula, a definition which implicitly includes the assumption of referencability.

In Explication 2.1 we gave a detailed definition of the assumptions to be made to the range of propositions which can be reasoned about in the field of an arbitrary propositional logic. As stated before the semantics of the class of $\in\mu$-logics fully reflects these assumptions which we want to show in the following theorem:

**Theorem 5.1** (Compliance with assumptions on propositions in logics). The interpretation and treatment of propositions in the models of the logics of the class of $\in\mu$-logics complies with the assumptions on propositions in logics as stated in Explication 2.1.

**Proof.** We successively show that the assumptions of Explication 2.1 are fulfilled by the definition of $\in\mu$-logics:

1. **Referencability of propositions:** The sense function $\Gamma$ of every model of every $\in\mu$-logic interprets formulas as propositions given in the set of available propositions $M$ of the corresponding model. However, for complex formulas the interpretation must not be surjective, that means that not every proposition must be referred to by a complex formula. However, by assigning an arbitrary proposition $m \in M$ to a variable $x$ in an assignment function $\beta$ we guarantee that the proposition denoted by $x$ under $\beta$ is given as the proposition $m$. Thus we have shown that every proposition is referencable by a certain formula of $\in\mu$-logics.

2. **Comparability of propositions:** The class of $\in\mu$-logics offers the operator of propositional identity $\equiv$ which states that the connected formulas are being interpreted as the same proposition. As the sense function $\Gamma$ explicitly interprets formulas as propositions and due to the semantics of the operator of propositional identity we achieve a comparability of the propositions denoted by formal expressions which is beyond a pure comparison of the truth values.

3. **Partition into true and false propositions:** The concept of $\in\mu$-structures offers a set of available propositions $M$ which is parted into a set of true propositions $T_w$ and a set of false propositions $M \setminus T_w$ in every world $w$ of the currently considered possible worlds structure.

4. **Closure under means of expressions:** The truth properties and integration properties guarantee that for every operator of a given formula the semantics of the operator is carried over to the propositional layer in the sense of the truth values. Furthermore the interpretation functions demanded in the fourth contextual property of the Definition of $\in\mu$-structures explicitly demand that there also is a structural representation of formulas in the propositional layer.

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As discussed by Bab in [1] it is not reasonable to demand the existence of such interpretation functions for the integrated modal operators, too. This is due to the fact that the sense of a formula possibly $\phi$ does not solely depend on the sense of the formula $\phi$ in the currently considered world, but on the sense of $\phi$ in all accessible worlds. Nevertheless, the definition of the integration properties for the integrated modal operators already demands for a consideration of the modal operators in a formula in the interpretation of this formula as a proposition.
5. **Consistency of referencability:** The interpretation functions demanded for the local operators in the fourth contextual property of the definition of $\in\mu$-structures result in a sense preserving (that means proposition preserving) exchangeability of subformulas in certain given formulas.\(^{24}\)

6. **Universal quantification:** The class of $\in\mu$-logics allows for the quantification over all available propositions by the universal quantifier $\forall$.

The above theorem shows that the class of $\in\mu$-logics reflects our understanding of the range of propositions which can be reasoned about in a *propositional logic*. Assumed that one accepts the assumptions given in Explication 2.1 to be valid the above theorem shows that the class of $\in\mu$-logics and their models form a theory of propositions.

## 6 Conclusion

The class of $\in\mu$-logics as introduced in this paper is a class of special *propositional logics* in which – contrary to classical logics – the variables of the $\in\mu$-logics are explicitly assigned to propositions. In classical logics, however, variables are referring to propositions only on a meta-level and the assignment of variables is limited to the truth-values *true* and *false*. In this understanding one can argue that there is no explicit reference between the written form of a formula and its intended sense in the semantics of classical logics. The same holds for predicate logics in which there indeed is an explicit definition of the predicates, but at the end this definition is not present in the interpretation of formulas, as formulas again are semantically interpreted only as *true* or *false*. In $\in\mu$-logics we have a completely different approach. Here the senses meant for the variables have to be made explicit by assigning the variables to propositions. Based on that fact the interpretation of complex formulas can yield to more complex propositions respectively propositions representing the intended sense of the complex formulas.

We have studied the notion of propositions and argued that a formal explicit treatment of propositions in a logic must fulfill a certain set of assumptions. We observed that these assumptions are fully reflected in the syntax and semantics of $\in\mu$-logics. This is due to the fact that in $\in\mu$-logics there is an explicit correlation between the written form of a formula and the proposition denoted by that formula in the semantics. The nature of propositions is not determined by the semantics of $\in\mu$-logics, but what is determined is a correlation between written formal expressions and their denoted propositions whereat there is a compliance with the semantics of the operators included in the formal expressions. This correlation between the written form and the denoted propositions allows for seeing the class of $\in\mu$-logics as a certain theory of propositions.

\(^{24}\)Note that due to Bab in [1] this exchangeability in the case of modal operators is only possible if the subformulas to be exchanged do denote the same sense in *every* available world.
References


